Solution 6

Supplementary Problems

1. Find the volume of the ball $x^2+y^2+z^2+w^2 \le R^2$ in $\mathbb{R}^4$ by the formula

$$\text{vol} = \int_{-R}^{R} |B_w|\, dw \ ,$$

where $|B_w|$ is the volume of the cross section of the ball at height w. The answer is $\pi^2 R^4/2$.

Solution. The cross section is a three dimensional ball of radius $\sqrt{R^2-w^2}$. Using this formula, the volume of the four dimensional ball is

$$\begin{aligned}
\int_{-R}^{R} |B_w|\, dw &= 2\int_0^R |B_w|\, dw \\
&= 2\int_0^R \frac{4\pi}{3}(R^2-w^2)^{3/2}\, dw \\
&= \frac{8\pi}{3}R^4 \int_0^{\pi/2} \sin^4\theta\, d\theta \quad (w = R\cos\theta) \\
&= \frac{\pi^2}{2}R^4
\end{aligned}$$

2. Let D be a region in the plane which is symmetric with respect to the origin, that is, $(x,y)\in D$ if and only if $(-x,-y)\in D$. Show that

$$\iint_D f(x,y)\, dA(x,y) = 0 \ ,$$

when f is odd, that is, $f(-x,-y) = -f(x,y)$ in D. This problem appears in Ex 4. Now you are asked to apply the change of variables formula in two dimension.

Solution. Just because the Jacobian of the map $(x,y)\mapsto(-x,-y)$ is equal to 1.

3. The rotation by an angle θ in anticlockwise direction is given by $(x,y) = (\cos\theta\ u - \sin\theta\ v, \sin\theta\ u + \cos\theta\ v)$. Verify that rotation leaves the area unchanged.

Solution. Let G be a region in the plane. The area of G is defined to be $\iint_G 1\, dA(u,v)$. After the rotation G to D, and the area of D is $\iint_D 1\, dA(x,y)$. The Jacobian of the change of variables $\dfrac{\partial(x,y)}{\partial(u,v)}$ is easily calculated to be 1. Therefore,

$$|D| = \iint_D 1\, dA(x,y) = \iint_G 1\times 1\, dA(u,v) = |G| \ .$$

Note. It is easy to verify that other Euclidean motions such as translations and reflections also leave the area unchanged. Their Jacobians are all equal to 1 or -1.

4. Consider the map $(u,v)\mapsto(x,y) = (u^2,v)$ which maps the square $R_1 = [-1,1]\times[0,1]$ onto $R_2 = [0,1]\times[0,1]$. Show that

$$\iint_{R_2} f(x,y)dA(x,y) \neq \iint_{R_1} f(u^2,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA(u,v) \ .$$

(Hint: It suffices to take $f(x, y) \equiv 1$.) Why?

Solution. On one hand,

$$\iint_{R_2} dA(x,y) = \int_0^1 \int_0^1 dxdy = 1 \ .$$

On the other hand, the Jacobian determinant for this map is $2u$. Letting R_3 to be the part of R_1 on the right and R_4 the part on the left, both of which are unit squares.

$$\begin{aligned}\iint_{R_1} |2u|\, dA(u,v) &= \iint_{R_3} |2u|\, dA(u,v) + \iint_{R_4} |2u|\, dA(u,v) \\ &= \int_0^1 \int_0^1 2u\, dvdu + \int_{-1}^0 \int_0^1 (-2u)\, dvdu \\ &= 2 \ .\end{aligned}$$

The change of variables formula is not valid because this map is not one to one. In fact, it is two to one.